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The probability distribution of the spectral form factor in random matrix theory

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Abstract. We determine the probability distribution of the spectral form factor from random matrix theory in the orthogonal and unitary case. We show that it is an exponential one, parametrized by the average value of this quantity.

1. Introduction

A quantity of central interest in the study of random matrices and their application to problems of quantum chaos has been the spectral form factor. $S_N(k)$ defined as

$$S_N(k) = \frac{1}{N}\left|\sum_{j=i}^{N} \mathrm{e}^{\mathrm{i}k\lambda j}\right|^2$$

the λj being N eigenvalues of a deterministic or random Hamiltonian.

Usually, after some unfolding of the eigenvalues has been made, so that their average distance is fixed to be 1, one computes either its spectral average, or some other ensemble average and compares its value to the average value given by random matrix theory. But the question arises to know if this average value is representative of the sample considered. If it is so, one says in the physics literature that this quantity is self-averaging.

Nuclear data collected by Bohigas [1], for example, show that the average value describes only the mean trend of this quantity and that large fluctuations around it are observed. A similar behaviour is observed in numerical results for the form factor of the hydrogen atom in a strong magnetic field [2].

As recently emphasized by Prange [3], these results point to the fact that the spectral form factor is not self-averaging. This conclusion was also reached previously by other authors [4–6]. The question then naturally arises of how to determine the probability distribution of the form factor. Argaman *et al* [6] used a semi-classical argument to conclude that this distribution should be exponential. Prange [3] reached the same conclusion by using a random walk analogy.

In the context of random matrix theory, this is a well-posed problem and the purpose of this paper is to prove rigorously that random matrix theory (in the orthogonal and unitary case at least) does indeed predict an *exponential distribution* for the form factor. The form factor is therefore obviously not self-averaging, but its probability distribution is parametrized by one

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quantity only, its average value. This shows that the computation of the average value is really the basic quantity to compute.

It would be, of course, quite interesting to see if this result of random matrix theory explains the dispersion of the experimental data observed by Bohigas [1].

Let us note, first, that we need to keep the parameter k in $S_N(k)$ strictly positive in order to have a well-posed problem, since when $k = 0$ there is no limiting distribution. Moreover, we will work with the ensemble which is technically the simplest one, namely the circular ensemble. This is justified from the known fact that after unfolding all correlations become universal. Therefore, the result should be unchanged if we take, for example, Gaussian ensembles.

The strategy used to obtain the probability distribution is the following: we compute the generating function of the random variables

$$\left(\frac{1}{\sqrt{N}}\sum_{j=1}^{N}\cos kN\theta_j, \frac{1}{\sqrt{N}}\sum_{j=1}^{N}\sin kN\theta_j\right)$$

θ_j being the eigenvalue of the circular ensemble.

This generating function is written in the form

$$G = \left\langle \prod_{j=1}^{N} u(\theta_j)\right\rangle$$

where

$$u(\theta) = \exp\left[\frac{2\mathrm{i}}{\sqrt{N}}\,\mathrm{Re}\, u\, \mathrm{e}^{\mathrm{i}kN\theta}\right]$$

u being a complex number. This quantity is expressed in the orthogonal case by means of the square root of a determinant and in the unitary case by means of a determinant. These determinants are of the form det $[1 + T]$, T being some matrix.

Then, using the identity

$$\det(1+T) = \exp\left[\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\,\mathrm{tr}\, T^n\right]$$

we would like to show that, when N tends to infinity, only the first two terms in this expansion remain, namely

$$\mathrm{tr}\, T - \tfrac{1}{2}\,\mathrm{tr}\, T^2$$

which are then computed explicitly.

In the unitary case, this can be done straightforwardly. The orthogonal case is different. In this case, tr T^n is given by non-absolutely convergent integrals, and therefore the fact that it vanishes in the limit N tends to infinity, when $n \geqslant 3$, is the result of rather subtle cancellations. We tackle this problem by a set of 'renormalizations' of the matrix, to transform it to a more reasonable matrix.

In any case, the final results are that in both cases, the generation function is asymptotically Gaussian. This shows that

$$x = \frac{1}{\sqrt{N}}\sum_{j=1}^{N}\cos kN\theta_j \qquad y = \frac{1}{\sqrt{N}}\sum_{j=1}^{N}\sin kN\theta_j$$

are Gaussian random variables with a distribution proportional to

$$\exp\left[-\frac{1}{2\sigma}(x^2+y^2)\right]$$

and this shows that the probability distribution of the form factor $S_N(k)$ is *exponential*.

More technically,

$$\lim_{N\to\infty} \mathrm{pr}\{S_N(k) \leqslant x\} = \int_0^x \frac{\mathrm{d}y}{s} \exp\left[-\frac{y}{s}\right]$$

where s is the average value of $S_N(k)$, in the large N limit. It is given (in Mehta's notation) by

$$s = 1 - b(k)$$

where $b(k)$ is the Fourier transform of the cluster two-point function.

The explicit forms of these functions are given in Mehta's book [7] and they are produced here at the end of sections 2 and 3.

2. The orthogonal case

The generating function can be written as

$$G = \left\langle \prod_{i=1}^{N} u(\theta_i) \right\rangle \tag{2.1}$$

with

$$\begin{aligned} u(\theta) &= \exp v_0(\theta) \\ v_0(\theta) &= \frac{\mathrm{i}u}{\sqrt{N}}\,\mathrm{e}^{\mathrm{i}kN\theta} + \frac{\mathrm{i}u^*}{\sqrt{N}}\,\mathrm{e}^{-\mathrm{i}kN\theta}. \end{aligned} \tag{2.2}$$

In Mehta's book [7], this quantity is expressed by means of a determinant (equations 10.4.5 and 10.4.6 of this reference)

$$G^2 = \det F \tag{2.3}$$

where if N is even, the $N \times N$ matrix f is given by

$$F_{p,q} = \frac{\mathrm{i}p}{4\pi} \int^{\pi}\!\!\int_{-\pi} u(\theta)u(\varphi)\sigma(\theta-\varphi)\,\mathrm{e}^{\mathrm{i}p\varphi - \mathrm{i}q\theta}\,\mathrm{d}\theta\,\mathrm{d}\varphi \tag{2.4}$$

with

$$\sigma(\theta) = \begin{cases} 1 & \text{if } \theta \geqslant 0 \\ -1 & \text{if } \theta < 0 \end{cases} \tag{2.5}$$

and

$$p, q = \frac{-N}{2} + \frac{1}{2}, -\frac{-N}{2} + \frac{3}{2}, \ldots, \frac{N}{2} - \frac{1}{2}. \tag{2.6}$$

When $u(\theta) = 1$, $G = 1$, so that we can write

$$G = [\det(1+T)]^{1/2}. \tag{2.7}$$

The matrix T is defined as

$$T_{p,q} = \frac{\mathrm{i}p}{4\pi} \int^{\pi}\!\!\int_{-\pi} [v(\theta) + v(\varphi) + v(\theta)v(\varphi)]\sigma(\theta-\varphi)\,\mathrm{e}^{\mathrm{i}p\varphi - \mathrm{i}q\theta} \tag{2.8}$$

where

$$v(\theta) = u(\theta) - 1. \tag{2.9}$$

Basically, we want to compute this determinant by using the formula

$$\det(1+T) = \exp\,\mathrm{tr}\,\ln(1+T) = \exp \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}\,\mathrm{tr}\,T^n \tag{2.10}$$

and show that tr T^n vanishes when $N \to \infty$, and $n \geqslant 3$. However, this results from rather subtle cancellations between the various terms appearing in tr T^n. We will arrive, therefore, at the result by 'renormalizing' the matrix T. For this purpose, we introduce the following three auxiliary matrices:

$$A_{p,q} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathrm{d}\theta\, \mathrm{e}^{+\mathrm{i}(p-q)\theta}\, v(\theta) \tag{2.11}$$

$$B_{p,q} = \frac{p}{q} A_{p,q} \tag{2.12}$$

$$C_{p,q} = \frac{\mathrm{i}p}{4\pi} \int^{\pi}\int_{-\pi} v(\theta)v(\varphi)\sigma(\theta-\varphi)\, \mathrm{e}^{\mathrm{i}p\varphi - \mathrm{i}q\theta}\, \mathrm{d}\theta\, \mathrm{d}\varphi \tag{2.13}$$

so that we have

$$T = A + B + C. \tag{2.14}$$

Therefore, we can also write the determinant as

$$\det(1+T) = [\det(1+A)][\det(1+B)][\det(1+T_1)] \tag{2.15}$$

with

$$T_1 = (1+B)^{-1} C (1+A)^{-1} - B(1+B)^{-1} A(1+A)^{-1} \tag{2.16}$$

assuming that the inverse of $1 + A$ and $(1 + B)$ exist (this will be proven later).

Since

$$\mathrm{tr}\, A^n = \mathrm{tr}\, B^n$$

we have

$$\det(1+A) = \det(1+B) \tag{2.17}$$

and

$$G = [\det(1+A)][\det(1+T_1)]^{1/2}. \tag{2.18}$$

We will show that tr T_1^n vanishes when $N \to \infty$, except for $n = 1$. The proof will be rather long and we need to introduce some useful tools.

On the space of bounded functions $f(\theta)$ with $\theta \in \Lambda = [-\pi, \pi]$, we define operators by their bounded kernel, in the usual way,

$$(Df)(\theta) = \int_{-\pi}^{\pi} D(\theta|\varphi) f(\varphi)\, \mathrm{d}\varphi \tag{2.19}$$

so that the kernel of the product of two operators D and F will be given by

$$(DF)(\theta|\varphi) = \int_{-\pi}^{\pi} \mathrm{d}\psi\, D(\theta|\psi) F(\psi|\varphi). \tag{2.20}$$

We will also introduce multiplication operators (always designated by small letters) as

$$(af)(\theta) = a(\theta) f(\theta) \tag{2.21}$$

so that the kernels of aD and Da are respectively

$$(aD)(\theta|\varphi) = a(\theta)D(\theta|\varphi) \qquad (Da)(\theta|\varphi) = D(\theta|\varphi)a(\varphi). \tag{2.22}$$

We will also call the trace of an operator

$$\mathrm{Tr}\, D = \int_{-\pi}^{\pi} \mathrm{d}\varphi\, D(\varphi|\varphi) \tag{2.23}$$

assuming that $D(\theta|\varphi)$ is continuous on the diagonal. Let us now introduce the operator S of kernel

$$S(\theta|\varphi) = \frac{1}{2\pi}\sum_q \mathrm{e}^{-\mathrm{i}q(\theta-\varphi)}. \tag{2.24}$$

It appears naturally when we compute A^n. Indeed, if $(vS)^0(\varphi|\theta) = \delta(\varphi-\theta)$

$$(A^n)_{p,q} = \frac{1}{2\pi}\int^{\pi}\int_{-\pi} \mathrm{d}\theta\,\mathrm{d}\varphi\,\mathrm{e}^{\mathrm{i}p\varphi-\mathrm{i}q\theta}(vS)^{n-1}(\varphi|\theta)v(\theta) \tag{2.25}$$

and of course

$$(B^n)_{p,q} = \frac{p}{q}(A^n)_{p,q}. \tag{2.26}$$

Moreover,

$$\operatorname{tr} A^n = \operatorname{Tr}(vS)^n. \tag{2.27}$$

Formula (2.25) is easily proved by induction by writing

$$(A^{n+1})_{p,q} = \sum_{q'}(A^n)_{p,q'}\,A_{q',p}$$

and inserting on the right-hand side the expression (2.25) for A^n and (2.11) for A, and using the definition (2.24) of $S(\theta|\varphi)$.

$(1+A)^{-1}$ is defined through its series when it is convergent, which is certainly the case if v is small enough.

Now, formally at least, we have

$$[A^2(1+A)^{-1}]_{p,q} = \frac{+1}{2\pi}\int^{\pi}\int_{-\pi}\mathrm{d}\theta\,\mathrm{d}\varphi\,\mathrm{e}^{\mathrm{i}p\varphi-\mathrm{i}q\theta}\left[-\sum_{n=2}^{\infty}(-vS)^{n-1}(\varphi|\theta)v(\theta)\right]. \tag{2.28}$$

In order to see if such series convergence and to estimate them, we now introduce some norms. We will call for an operator D with a bounded kernel

$$\begin{aligned}
|D| &= \sup_{(\theta,\varphi)\in\Lambda}|D(\varphi|\theta)| \\
|D|_+ &= \sup_{\varphi\in\Lambda}\int_{-\pi}^{\pi}|D(\varphi|\theta)|\,\mathrm{d}\theta \\
|D|_- &= \sup_{\theta\in\Lambda}\int_{-\pi}^{\pi}|D(\varphi|\theta)|\,\mathrm{d}\varphi
\end{aligned} \tag{2.29}$$

and for multiplication operators a

$$|a| = \sup_{\theta\in\Lambda}|a(\theta)|. \tag{2.30}$$

We will use repeatedly a certain number of properties of these norms. Their proof follows immediately from the definitions.

Property 1.

$$\begin{aligned}
|AB| &\leqslant 2\pi|A||B| \\
|AB| &\leqslant |A|_+|B| \\
|AB| &\leqslant |A||B|_- \\
|AB|_\pm &\leqslant |A|_\pm|B|_\pm
\end{aligned} \tag{2.31}$$

$$\begin{aligned}
&\sup(|aB|, |Ba|) \leqslant |a|\,|B| \\
&\sup(|aB|_\pm, |Ba|_\pm) \leqslant |a|\,|B|_\pm.
\end{aligned} \tag{2.32}$$

Consider now the operator S. We have

$$S(\theta|\varphi) = s(\theta - \varphi) \tag{2.33}$$

with

$$s(\theta) = \frac{1}{2\pi} \frac{\sin(N\theta/2)}{\sin(\theta/2)}. \tag{2.34}$$

From the inequality

$$|\sin\theta| \geqslant c|\theta| \tag{2.35}$$

which is valid if $\theta \in [0, \pi/2]$, where c is some constant, we see that we have

$$|S| = \mathrm{O}(N) \qquad |S|_{\pm} = \mathrm{O}(\ln N). \tag{2.36}$$

Now since

$$|v| = \mathrm{O}\left(\frac{1}{\sqrt{N}}\right) \tag{2.37}$$

we see by using property 1 that

$$|vS|^n = \mathrm{O}\left(\sqrt{N}\left(\frac{\ln N}{\sqrt{N}}\right)^{n-1}\right) \tag{2.38}$$

and recalling the expression appearing in formula (2.28)

$$\alpha(\varphi|\theta) = -\sum_{n=2}^{\infty}(-vS)^{n-1}(\varphi|\theta)v(\theta) \tag{2.39}$$

we see that

$$\begin{aligned} |\alpha| &= \mathrm{O}(1) \\ |\alpha|_{\pm} &= \mathrm{O}\left(\frac{\ln N}{N}\right) \end{aligned} \tag{2.40}$$

and that the series defining $(1 + A)^{-1}$ converges. We have, of course,

$$[B^2(1 + B)^{-1}]_{p,q} = \frac{p}{q}[A^2(1 + A)^{-1}]_{p,q}. \tag{2.41}$$

If we introduce the operator Σ of a kernel

$$\Sigma(\varphi|\theta) = \sigma(\varphi - \theta) \tag{2.42}$$

we can write

$$C_{p,q} = \frac{\mathrm{i}p}{4\pi} \int^{\pi}\int_{-\pi} [v\Sigma v](\varphi|\theta)\, \mathrm{e}^{\mathrm{i}p\varphi - \mathrm{i}q\theta}. \tag{2.43}$$

Using formulae (2.28), (2.41) and (2.43), and the definitions of A and B, we see that

$$[(1 + B)^{-1} C(1 + A)^{-1} - C]_{p,q} = \frac{\mathrm{i}p}{4\pi} \int^{\pi}\int_{-\pi} \mathrm{d}\theta\, \mathrm{d}\varphi\, \gamma(\varphi|\theta)\, \mathrm{e}^{\mathrm{i}p\varphi - \mathrm{i}q\theta} \tag{2.44}$$

with

$$\gamma = (v + \alpha)Sv\,\Sigma\, v + v\,\Sigma\, vS(v + \alpha) - (v + \alpha)Sv\,\Sigma\, vS(v + \alpha). \tag{2.45}$$

Using property 1 and the estimates (2.40), we get

$$|\gamma| = \mathrm{O}\left(\frac{(\ln N)^2}{N^{3/2}}\right). \tag{2.46}$$

In order to obtain a final expression for T_1, we need to compute

$$[B(1+B)^{-1}A(1+A)^{-1}]_{p,q} = \frac{\mathrm{i}p}{2\pi}\int^{\pi}\int_{-\pi} \mathrm{d}\theta\,\mathrm{d}\varphi[(v+\alpha)J(v+\alpha)](\varphi|\theta)\,\mathrm{e}^{\mathrm{i}p\varphi-\mathrm{i}q\theta} \tag{2.47}$$

where

$$J(\varphi|\theta) = \frac{1}{2\pi}\sum_q \frac{1}{\mathrm{i}q}\,\mathrm{e}^{-\mathrm{i}q(\varphi-\theta)} \tag{2.48}$$

which follows again from the fact that $B_{p,q} = (p/q)A_{p,q}$.

We have therefore shown that we can express T_1 as

$$(T_1)_{p,q} = \frac{\mathrm{i}p}{2\pi}\int^{\pi}\int_{-\pi} \mathrm{d}\theta\,\mathrm{d}\varphi\,\mathrm{e}^{\mathrm{i}p\varphi-\mathrm{i}q\theta}\,T_1(\varphi|\theta) \tag{2.49}$$

with

$$T_1 = -\tfrac{1}{2}v\,\Sigma\,v + \tfrac{1}{2}\gamma - (v+\alpha)J(v+\alpha). \tag{2.50}$$

Let us separate T_1 into two parts

$$T_1 = \Gamma_1 + R_1 \tag{2.51}$$

where

$$\Gamma_1 = -\tfrac{1}{2}v_0\cdot\Sigma\cdot v_0 - v_0\cdot J\,v_0. \tag{2.52}$$

The usefulness of this separation comes from the fact that

$$|v - v_0| = \mathrm{O}\left(\frac{1}{N}\right) \tag{2.53}$$

and therefore

$$|v\,\Sigma\,v - v_0\,\Sigma\,v_0| = \mathrm{O}\left(\frac{1}{N^{3/2}}\right). \tag{2.54}$$

Moreover, from the definition of the kernel of J, equation (2.48), we see that

$$|J| = \mathrm{O}(\ln N) \tag{2.55}$$

and therefore

$$|v\,J\,v - v_0\,J\,v_0| = \mathrm{O}\left(\frac{\ln N}{N^{3/2}}\right) \tag{2.56}$$

and using the estimate (2.40) and (2.55)

$$|v\,J\,\alpha + \alpha\,J\,v + \alpha\,J\,\alpha| = \mathrm{O}\left(\frac{(\ln N)^2}{N^{3/2}}\right). \tag{2.57}$$

Combining all these estimates with that in equation (2.46), we see that

$$|R_1| = \mathrm{O}\left(\frac{(\ln N)}{N^{3/2}}\right). \tag{2.58}$$

We need to find a useful expression for T_1^n. This is accomplished by the following identity

$$(T_1)^n_{p,q} = \frac{\mathrm{i}p}{2\pi}\int^{\pi}\int_{-\pi} \mathrm{d}\theta\,\mathrm{d}\varphi\,\mathrm{e}^{\mathrm{i}p\varphi-\mathrm{i}q\theta}[(T_1 I)^{n-1}\,T_1](\varphi|\theta) \tag{2.59}$$

where the operator I has the kernel

$$I(\theta|\varphi) = \frac{1}{2\pi}\sum_q \mathrm{i}q\,\mathrm{e}^{-\mathrm{i}q(\theta-\varphi)}. \tag{2.60}$$

We see therefore that

$$\operatorname{tr} T_1^n = \operatorname{Tr}(T_1 I)^n. \tag{2.61}$$

In order to proceed further we need therefore to analyse the operator $\tau = T_1 I$, decomposed as

$$\tau = \Gamma + R \tag{2.62}$$

with

$$\Gamma = \Gamma_1 I \qquad R = R_1 I. \tag{2.63}$$

From equation (2.60) it is clear that

$$|I| = \mathrm{O}(N^2) \tag{2.64}$$

but we can also note that

$$I(\theta|\varphi) = i(\theta - \varphi) \tag{2.65}$$

with

$$i(x) = -s'(x) = \frac{1}{2\pi}(N-1)\frac{\sin^2(Nx/4)}{\sin(x/2)} + \frac{1}{4\pi}\frac{\sin((N-1)x/2) - (N-1)\sin(x/2)}{\sin^2(x/2)}. \tag{2.66}$$

Using in this expression, inequality (2.35) and the inequality $|\sin x - N\ \sin(x/N)| \leqslant x^3/3$, one can show that

$$|I|_\pm = \mathrm{O}(N\ \ln\ N). \tag{2.67}$$

It follows from this that

$$|R| = \mathrm{O}\left(\frac{(\ln N)^3}{\sqrt{N}}\right) \tag{2.68}$$

$$|\Gamma| = \mathrm{O}((\ln N)^2). \tag{2.69}$$

Such estimates give

$$|\Gamma^2 - \tau^2| = \mathrm{O}\left(\frac{(\ln N)^5}{\sqrt{N}}\right). \tag{2.70}$$

In order to proceed further, we need to analyse in a more refined way the operator Γ, which we recall is given explicitly by

$$-\Gamma = \tfrac{1}{2} v_0\, \Sigma\, v_0\, I + v_0\, J\, v_0\, I. \tag{2.71}$$

From the definition given in equation (2.48), we have

$$J(\varphi|\theta) = -\int_0^{\varphi-\theta} \mathrm{d}t\, s(t). \tag{2.72}$$

We decompose this kernel in the following way

$$J(\varphi|\theta) = -\sigma(\varphi-\theta)\ell(0) + \sigma(\varphi-\theta)\ell(|\varphi-\theta|) \tag{2.73}$$

where the function ℓ is defined by

$$\ell(\theta) = \int_\theta^\pi s(t)\,\mathrm{d}t. \tag{2.74a}$$

To this decomposition there corresponds the following for the operators,

$$J = -\ell(0)\Sigma + L \tag{2.74b}$$

and we can rewrite Γ as

$$-\Gamma = [\tfrac{1}{2} - \ell(0)]v_0\, \Sigma\, v_0\, I + v_0\, L\, v_0\, I. \tag{2.75}$$

An integration by parts allows us to express the function $\ell(\theta)$ as

$$\ell(\theta) = \frac{1}{\pi N}(1 - \cos(\pi N/2)) - \frac{1}{\pi N}\frac{(1 - \cos(N\theta/2))}{\sin(\theta/2)} + \frac{1}{\pi N}\int_{\theta/2}^{\pi/2} \frac{1 - \cos Nt}{(\sin t)^2} \cos t\, \mathrm{d}t. \tag{2.76}$$

This shows that

$$\ell(0) = \frac{1}{2} + \mathrm{O}\left(\frac{1}{N}\right) \tag{2.77}$$

where we have used the fact that

$$\int_0^\infty \frac{1 - \cos t}{t^2} = \frac{\pi}{2}. \tag{2.78}$$

Therefore, the first term in Γ can be estimated as

$$\left|\left[\frac{1}{2} - \ell(0)\right]v_0\, \Sigma\, v_0\, I\right| = \mathrm{O}\left(\frac{\ln N}{N}\right). \tag{2.79}$$

Using the expression of $\ell(\theta)$ given by equation (2.76) in the range $0 \leqslant \theta \leqslant \pi$, and the fact that $\ell(2\pi - \theta) = \ell(\theta)$, as well as the estimate $|x\, i(x)| = \mathrm{O}(N)$, we can obtain the following properties for the last term in Γ

$$|v_0\, L\, v_0\, I| = \mathrm{O}(\ln N) \tag{2.80}$$

$$|v_0\, L\, v_0\, I|_\pm = \mathrm{O}\left(\frac{(\ln N)^2}{2}\right) \tag{2.81}$$

from which we conclude that

$$|\Gamma^2| = \mathrm{O}\left(\frac{(\ln N)^3}{N}\right) \tag{2.82}$$

and therefore

$$|\tau^2| = \mathrm{O}\left(\frac{(\ln N)^5}{\sqrt{N}}\right). \tag{2.83}$$

This shows that

$$\left|\sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{n} \operatorname{tr} T_1^n\right| = \mathrm{O}\left(\frac{(\ln N)^7}{\sqrt{N}}\right) \tag{2.84}$$

and therefore, combining all these estimates,

$$\det(1 + T_1) = \exp\left[-\operatorname{tr} v_0\, L\, v_0\, I + \mathrm{O}\left(\frac{(\ln N)^7}{\sqrt{N}}\right)\right]. \tag{2.85}$$

It remains to compute $\det(1 + A)$. Since

$$\operatorname{tr} A^n = \operatorname{Tr}(vS)^n \tag{2.86}$$

and

$$|(vS)^n| = \mathrm{O}\left(\left(\frac{(\ln N)}{\sqrt{N}}\right)^{n-1}\sqrt{N}\right) \tag{2.87}$$

we can also conclude that

$$\det(1 + A) = \exp\left[\operatorname{tr}(vS) - \frac{1}{2}\operatorname{tr}(vS)^2 + \mathrm{O}\left(\frac{(\ln N)^2}{\sqrt{N}}\right)\right]. \tag{2.88}$$

Finally, using the expression for the generating function in terms of determinants (equation (2.18)), we obtain the desired result

$$G = \exp\left\{\frac{+1}{2}\mathrm{Tr}[v_0^2\, S - (v_0\, S)^2 - v_0\, L\, v_0\, I]\right\} \tag{2.89}$$

with an accuracy of at least $((\ln N)^7/\sqrt{N})$.

It remains to compute these traces explicitly in the limit where N goes to infinity. We have

$$-\mathrm{Tr}\, v_0\, S = 2|u|^2 + \mathrm{O}\left(\frac{1}{N}\right) \tag{2.90}$$

$$-\mathrm{Tr}(v_0\, S)^2 = \frac{8\pi\,|u|^2}{N}\int_0^\pi \cos(kN\,\theta)\, s^2(\theta)\,\mathrm{d}\theta + \mathrm{O}\left(\frac{\ln N}{N}\right) \tag{2.91}$$

and

$$\begin{aligned}-\mathrm{Tr}\, v_0\, L\, v_0\, I = {}& \frac{-4|u|^2}{N} s(0)\,\ell(0) + \mathrm{Tr}(v_0\, S)^2 \\ & +4|u|^2\, k\int_0^\pi \sin(kN\,\theta)\, s(\theta)\,\ell(\theta)\,\mathrm{d}\theta + \mathrm{O}\left(\frac{\ln N}{N}\right).\end{aligned} \tag{2.92}$$

In order to derive the last expression, we used the act that $i(\theta) = -s'(\theta)$ and $\ell'(\theta) = -s(\theta)$.

Now all the integrals appearing in these expressions have a limit given by an absolutely convergent integral, if we use for $\ell(\theta)$ the expression given in equation (2.76).

The corresponding integrals can be computed explicitly and are given by

$$\lim_{N\to\infty}\frac{1}{N}\int_0^\pi \cos(kN\,x)s^2(x)\,\mathrm{d}x = \frac{1}{4\pi}(1-|k|)\theta(1-|k|) \tag{2.93}$$

and

$$\lim_{N\to\infty} k\int_0^\pi \sin(kN\,x)\ell(x)s(x)\,\mathrm{d}(x) = \frac{|k|}{4\pi}[\ln(2|k|+1) - \theta(|k|-1)\ln(2|k|-1)] \tag{2.94}$$

where $\theta(x)$ is the heaviside step function.

We have therefore obtained, as announced, a Gaussian distribution for the generating function

$$\lim_{N\to\infty} G(u) = \exp[-|u|^2(1-b(k))] \tag{2.95}$$

where $b(k)$ is the well known [47] Fourier transform of the two-point cluster function

$$b(k) = 1 - 2|k| + |k|\ln(1+2|k|)$$

if $|k| \leqslant 1$ and

$$b(k) = -1 + |k|\ln\frac{2|k|+1}{2|k|-1} \tag{2.96}$$

if $|k| \geqslant 1$.

This gives for the form factor

$$S(k) = \frac{1}{N}\left|\sum_{j=1}^{N} \mathrm{e}^{\mathrm{i}kN\theta_j}\right|^2$$

an exponential distribution, with mean $1 - b(k)$.

3. The unitary case

In this case, the generating function

$$G = \left\langle \sum_{j=1}^{N} u(\theta_j) \right\rangle \tag{3.1}$$

is computed with respect to a probability distribution proportional to

$$|\Delta|^2 = \left| \prod_{N \geqslant k \geqslant \ell \geqslant 1} (e^{i\theta_k} - e^{i\theta_\ell}) \right|^2 \tag{3.2}$$

which can be written as the modulus square of a Van der Monde determinant [7]

$$|\Delta|^2 = |\det \Lambda|^2 \tag{3.3}$$

where

$$\Lambda_{kj} = e^{i(k-1)\theta_j}. \tag{3.4}$$

Since

$$\det \Lambda = \sum_P (-1)^P \sum_{j=1}^{N} e^{i\theta_j (Pj-1)} \tag{3.5}$$

P being a permutation, we see easily by writing $|\det \Lambda|^2$ as a double sum over permutations, using (3.5), that

$$G = c \det \left[\int_{-\pi}^{\pi} u(\theta)\, e^{i\theta(k-j)} \right] \tag{3.6}$$

where c is some constant, which has to be 1 when $u(\theta) = 1$.

Introducing the same labelling as in the orthogonal case, when N is even, we get

$$G = \det(1 + A) \tag{3.7}$$

A being the matrix already introduced,

$$A_{p,q} = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\theta\, e^{i(p-q)\theta}\, v(\theta). \tag{3.8}$$

It follows therefore from the results previously established that asymptotically

$$G = \exp\{\tfrac{1}{2} \operatorname{Tr}[v_0^2 S - (v_0 S)^2]\} \tag{3.9}$$

and therefore, using equations (2.90), (2.91) and (2.94),

$$\lim_{N\to\infty} G = \exp\{-|u|^2[1 - b(k)]\} \tag{3.10}$$

with

$$b(k) = (1 - |k|)\theta(1 - |k|) \tag{3.11}$$

the Fourier transform of the two-point cluster function and this implies again for the form factor an exponential distribution with mean $1 - b(k)$.

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